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Completeness of Integer Translates in Function Spaces on $\mathbb{R}$

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We show that each of the Banach spaces $C_0(\mathbb{R})$ and $L^p(\mathbb{R})$, $2<p<\infty$, contains a function whose integer translates are complete. This function can also be chosen so that one of the following additional conditions hold: (1) Its non-negative integer translates are already complete. (2) Its integer translates form an orthonormal system in $L^2(\mathbb{R})$. (3) Its integer translates form a minimal system. A similar result holds for the corresponding Sobolev space, for certain weighted L^2 spaces, and in the multivariate setting. We also prove some results in the opposite direction.

1. INTRODUCTION

Assume that B is a natural translation invariant Banach space of functions on $\mathbb{R}$, such as $C_0(\mathbb{R})$ or $L^p(\mathbb{R})$, $1 \leqslant p < \infty$. Does it contain a function φ whose integer translates φ_n (defined as usual by $\varphi_n(x) = \varphi(x-n)$, $x \in \mathbb{R}$) are complete; that is, their linear combinations are dense in B?

Since the Fourier transform of every such linear combination is of the form $u\hat{\varphi}$, where u is a trigonometric polynomial with integer frequencies and $\hat{\varphi}$ is the Fourier transform of φ, a heuristic argument suggests that the Fourier transform of every element in the closed linear span of these translates is of the form $h\hat{\varphi}$, where h is a function of period 2π, and therefore, it will not contain any non-integer translate of φ. Thus one is led to believe that the answer to the question is negative. It is therefore somewhat surprising that it is in fact positive for some of the classical Banach spaces of functions on $\mathbb{R}$, such as $C_0(\mathbb{R})$, $L^p(\mathbb{R})$, $2<p<\infty$, the corresponding Sobolev spaces, and some weighted L^2 spaces. We prove that each of these contains a function φ whose integer translates are complete. Moreover, φ can be chosen in $C_0(\mathbb{R}) \cap L^2(\mathbb{R})$ and as the restriction to $\mathbb{R}$ of an entire function of order 1, with one of the following additional properties.

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(1) The non-negative integer translates $\{\varphi_n, n=0, 1, \ldots\}$ are already complete.

(2) $\{\varphi_n, n \in \mathbb{Z}\}$ is an orthonormal system in $L^2(\mathbb{R})$.

(3) $\{\varphi_n, n \in \mathbb{Z}\}$ is a minimal system, that is, non of its elements belongs to the closed linear span of the others.

The situation for the spaces $L^p(\mathbb{R})$, $1 \leqslant p \leqslant 2$, is completely different. They contain no finite number of elements such that the linear span of their integer translates is dense. In fact, we show in Section 4 that the same is true for most of the Banach spaces of tempered distributions, such that the Fourier transforms of their elements are functions.

An extensive literature is devoted to the study of completeness of all translates of a function in $L^p(\mathbb{R})$ (cf. [4, 7, 13, 20, 23, 26]). It is known that the set of all translates of a function in $L^p(\mathbb{R}) \cap L^1(\mathbb{R})$, whose Fourier transform has no zeros, is complete in $L^p(\mathbb{R})$ for $1 \leqslant p < \infty$ (cf. [4, 23]). For $p=1$, this is the celebrated general Tauberian theorem of Wiener [26].

For some results on the completeness of translates in $C_0(\mathbb{R})$, we refer to [2] and [8]. Completeness of integer translates of a function in $L^p(\mathbb{R})$ in certain subspaces associated with it is studied in [11] and [12]. The closed spans of (multi) integer translates of finitely many elements in $L^2(\mathbb{R}^n)$ are characterized in [6].

In Section 2, we state our main result and show that for a wide class of Banach spaces of functions on $\mathbb{R}$, including the spaces described in the first paragraph, the existence of a function with complete integer translates can be reduced to the existence of certain sets of uniqueness associated with the dual space.

In Section 3, we establish the existence of such sets of uniqueness for the duals of the above-mentioned spaces and thereby complete the proof of our main result. This requires the construction of certain countably infinite partitions of the interval $[-\pi, \pi]$, whose members are measurable sets that satisfy appropriate conditions. The condition required to prove the result for $C_0(\mathbb{R})$ and the related spaces of differentiable functions is that every member of the partition should have intersection of positive measure with every open interval included in $[-\pi, \pi]$. The existence of such a partition is known and elementary. The condition required to prove the result for $L^p(\mathbb{R})$, $2 < p < \infty$, and the corresponding Sobolev spaces is that every member of the partition should be a set of uniqueness for the space of functions in $L^1(\mathbb{T})$ whose Fourier coefficients are in $l^q(\mathbb{Z})$, where $q = p(p-1)^{-1}$. Sets of this type were constructed for the first time in [15] and [20]. Related results appear in [10] and [21]. The condition required to prove the result for the weighted L^2 spaces is of a similar nature.

In Section 4, we prove some results in the opposite direction. We show that the integer translates of a function in $L^1(\mathbb{R})$ are not complete in any of the spaces considered in our main theorem. We also prove that every translation invariant Banach space of tempered distributions, which is mapped continuously by the Fourier transform into the Fréchet space $L^1_{\mathrm{loc}}(\mathbb{R})$ and which contains an element whose Fourier transform is different from zero a.e. is not the closed span of integer translates of finitely many elements.

We deduce from this fact that none of the Banach spaces described in our main result has a Schauder basis which consists of integer translates of a single element.

In Section 5, we note that all our results can be extended to the multivariate setting and prove that the Banach space $C_0[0, 1]$ (of all functions in $C[0, 1]$ that vanish at 0 and 1) contains a function, such that the sequence obtained by composing it with all iterates of the function $x \to x^2$ is complete.

This paper originated from a problem which arose in a joint research with Gilles Cassier on the multiplicity of direct sums of unitary operators on Banach spaces.

2. SINGLY GENERATED SPACES

In this section, we formulate our main result and present in a general setting the functional analytic part of its proof. We begin with some notations and definitions.

In what follows, we shall denote by $C_0(\mathbb{R})$ the Banach space of all continuous functions f on $\mathbb{R}$ such that

$$\lim_{x \to \pm\infty} f(x) = 0,$$

equipped with the maximum norm. For every non-negative integer k, we denote by $C_0^k(\mathbb{R})$ the Banach space of all functions in $C_0(\mathbb{R})$ whose first k derivatives are also in this space, with norm defined as the sum of maximum norms of the function and its k derivatives, and for $1 \leqslant p < \infty$, we denote by $H^{p,k}(\mathbb{R})$ the Sobolev space of all functions in $L^p(\mathbb{R})$, whose first k distributional derivatives are also in $L^p(\mathbb{R})$, equipped with the usual norm (see [25, p. 323]). Thus $C_0^0(\mathbb{R}) = C_0(\mathbb{R})$, and $H^{p,0}(\mathbb{R}) = L^p(\mathbb{R})$. We shall use the notation $C_0^\infty(\mathbb{R})$ for $\bigcap_{k=0}^\infty C_0^k(\mathbb{R})$, and $H^{2,\infty}(\mathbb{R})$ for $\bigcap_{k=0}^\infty H^{2,k}(\mathbb{R})$. As will be noted in the sequel, $H^{2,\infty}(\mathbb{R}) \subset C_0^\infty(\mathbb{R})$.

We recall (cf. [17, p. 9]) that a continuous convex function M: $[0, \infty) \to [0, \infty)$ is called an N-function, if

$$\lim_{t \to 0+} \frac{M(t)}{t} = 0 \qquad \text{and} \qquad \lim_{t \to \infty} \frac{M(t)}{t} = \infty.$$

For such a function, we shall denote by $E_M(\mathbb{C})$, the vector space of the entire functions F on $\mathbb{C}$ such that

$$\sup_{z \in \mathbb{C}} |F(z)| \exp(-M(|z|)) < \infty.$$

Following [14], we shall say that a positive function ρ on $\mathbb{R}$ is a tempered weight function, if there exist positive constants c and r, such that

$$\rho(x+y) \leqslant c(1+|x|)^r \rho(y), \qquad x, y \in \mathbb{R}.$$

As shown in [14, p. 34], every such function is continuous. Typical examples of tempered weight functions are $\rho(x) = (1+|x|)^\alpha$ and $\rho(x) = [\log(1+|x|)]^\alpha$, where α is a real number.

For a function f on $\mathbb{R}$, and a real number y we shall denote by f_y the translation of f by y, that is, the function

$$f_y(x) = f(x-y), \qquad x \in \mathbb{R}.$$

If B is a Banach space of functions on $\mathbb{R}$, we shall say that it is singly generated, if it contains a function φ whose integer translates $\{\varphi_n, n \in \mathbb{Z}\}$ are complete, that is, their linear span is dense in B. Such a function will be called a generator of B.

We are now in a position to state our main result.

THEOREM 2.1. *Each of the Banach spaces* $C_0^k(\mathbb{R})$, $H^{p,k}(\mathbb{R})$, $2 < p < \infty$, $k = 0, 1, \ldots$, *and* $L^2(\mathbb{R}, \rho(x)\,dx)$, *where* ρ *is a tempered weight function in* $C_0(\mathbb{R})$, *is singly generated. Moreover, for every N-function* M, *each of these spaces has a generator* φ *which is contained in* $H^{2,\infty}(\mathbb{R})$, *and is the restriction to* $\mathbb{R}$ *of a function in* $E_M(\mathbb{C})$, *and also satisfies one of the following conditions*:

(1) *The translates* $\{\varphi_n, n = 0, 1, \ldots\}$ *are already complete.*

(2) $\{\varphi_n, n \in \mathbb{Z}\}$ *is an orthonormal system in* $L^2(\mathbb{R})$.

(3) $\{\varphi_n, n \in \mathbb{Z}\}$ *is a minimal system.*

Remark. By taking the function M above to be

$$M(x) = x \log(x+1), \qquad x \geqslant 0,$$

we see that φ can be chosen to be the restriction to $\mathbb{R}$ of an entire function of order 1. As we shall note in Section 4, φ cannot be chosen to be of finite exponential type.

The proof of the theorem is carried out in several steps. First we establish some general results which reduce its conclusion to the existence of certain measurable subsets of $\mathbb{R}$, which are associated with these spaces. In order to deal with all of them simultaneously, we have to single out a certain class of Banach spaces.

DEFINITION 1. If B is a Banach space of locally integrable functions on $\mathbb{R}$, and k is a non-negative integer, we shall say that B is of class $\mathscr{H}_k$ if it includes the Banach space $H^{2,k}(\mathbb{R})$ and the embedding is continuous and dense. We shall say that B is of class $\mathscr{H}$ if it is of class $\mathscr{H}_k$ for some k.

All the spaces described in the theorem are of class $\mathscr{H}$. It is clear that the spaces $L^2(\mathbb{R}, \rho(x)\,dx)$, where ρ is a tempered weight function in $C_0(\mathbb{R})$, are of class $\mathscr{H}_0$. For $k=0, 1, ...,$ and $2<p<\infty$, the spaces $C_0^k(\mathbb{R})$ and $H^{p,k}(\mathbb{R})$ are of class $\mathscr{H}_{k+1}$. This can be seen as follows. The Fourier transform maps the Banach space $L^1(\mathbb{R}, (1+|x|)^k\,dx)$ continuously into $C_0^k(\mathbb{R})$ (see [16, p. 123]) and maps the Banach space $L^2(1+x^2)^{k+1}\,dx)$ isometrically onto $H^{2,k+1}(\mathbb{R})$ (see [25, Proposition 31.6]). Therefore, since by the Schwarz inequality, $L^2(\mathbb{R}, (1+x^2)^{k+1}\,dx)$ is continuously imbedded in $L^1(\mathbb{R}, (1+|x|)^k\,dx)$, we get that $H^{2,k+1}(\mathbb{R})$ is continuously imbedded in $C_0^k(\mathbb{R})$. Since for $p>2$

$$C_0^k(\mathbb{R}) \cap H^{2,k}(\mathbb{R}) \subset H^{p,k}(\mathbb{R}),$$

we obtain that $H^{2,k+1}(\mathbb{R})$ is also continuously imbedded in $H^{p,k}(\mathbb{R})$. These embeddings are dense, since $\mathscr{D}(\mathbb{R})$ (the space of C^∞ functions with compact support on $\mathbb{R}$) is dense in all of these spaces [25, Proposition 37.5]. It also follows from these inclusions that

$$H^{2,\infty}(\mathbb{R}) \subset C_0^\infty(\mathbb{R}).$$

We establish next a useful criterion for a function in $H^{2,k}(\mathbb{R})$ to be a generator of a given Banach space of class $\mathscr{H}_k$. We make first some preliminary observations and introduce several notations.

Assume that B is a Banach space of class $\mathscr{H}_k$. Since the embedding of $H^{2,k}(R)$ in B is continuous and dense, its adjoint embeds the dual space B^* continuously into the dual space of $H^{2,k}(\mathbb{R})$, which is the Sobolev space $H^{2,-k}(\mathbb{R})$ [25, Proposition 31.2]. We recall that $H^{2,-k}(\mathbb{R})$ consists of all tempered distributions v, whose Fourier transform $\hat{v}$ belongs to $L^2(\mathbb{R}, (1+x^2)^{-k}\,dx)$, and the norm of v is defined as the norm of $\hat{v}$ in the latter space. The duality between $H^{2,k}(\mathbb{R})$ and $H^{2,-k}(\mathbb{R})$ is implemented by the pairing

$$\langle u, v\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(t)\hat{v}(-t)\,dt, \qquad u \in H^{2,k}(\mathbb{R}), \quad v \in H^{2,-k}(\mathbb{R}).$$

We observe that if u is in $H^{2,k}(\mathbb{R})$, and v is in $H^{2,-k}(\mathbb{R})$, then $\hat{u}$ is in $L^2(\mathbb{R}, (1+x^2)^k\,dx)$, and $\hat{v}$ is in $L^2(\mathbb{R}, (1+x^2)^{-k}\,dx)$, and therefore the function

$$t \to \hat{u}(t)\hat{v}(-t), \qquad t \in \mathbb{R},$$

is in $L^1(\mathbb{R})$, so that the above integral is well defined. By the previous observations, this formula remains true for u in B and v in B^* if we denote by $\langle\ ,\ \rangle$ also the pairing which implements the duality between B and B^*. We shall use this fact freely in the sequel without further mention.

In what follows, we adopt the standard notations, $\mathbb{T}$ for the one dimensional torus (identified in the usual way with the quotient group $\mathbb{R}/2\pi\mathbb{Z}$), and $L^p(\mathbb{T})$, $1 \leqslant p \leqslant \infty$, for the Banach space of 2π-periodic measurable fucntions g on $\mathbb{R}$, for which the norm

$$\|g\|_{L^p(\mathbb{T})} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(t)|^p\,dt\right)^{1/p}$$

is finite. For a function g in $L^1(\mathbb{T})$, we shall denote for every integer n, by $c_n(g)$ its nth Fourier coefficient, i.e.,

$$c_n(g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\,e^{-int}\,dt, \qquad n \in \mathbb{Z}.$$

For a function f in $L^1(\mathbb{R})$, we shall denote by Pf its 2π-periodization, that is, the function on $\mathbb{R}$ defined by

$$Pf = \sum_{j=-\infty}^{\infty} f_{2\pi j}.$$

We recall (cf. [16, p. 128]) that this series converges absolutely a.e. on $\mathbb{R}$, represents a function in $L^1(\mathbb{T})$, and the relation between its Fourier coefficients and the Fourier transform of f is given by

$$c_n(Pf) = (2\pi)^{-1}\,\hat{f}(n), \qquad n \in \mathbb{Z}.$$

For a function h on $\mathbb{R}$, we shall denote in the sequel by $\tilde{h}$, the function on $\mathbb{R}$ defined by

$$\tilde{h}(x) = h(-x), \qquad x \in \mathbb{R}.$$

The criterion alluded to before is given by

PROPOSITION 2.2. *Let k be a non-negative integer, and assume that B is a Banach space of class $\mathcal{H}_k$. A necessary and sufficient condition for a function*

φ in $H^{2,k}(\mathbb{R})$ to be a generator of B, is that the only element v in B^* for which

$$P(\tilde{\hat{\varphi}}\hat{v}) = 0$$

is the zero element.

Proof. Let φ be in $H^{2,k}(\mathbb{R})$, and assume that B is a Banach space of class $\mathcal{H}_k$. By the Hahn–Banach theorem, a necessary and sufficient condition for φ to be a generator of B is that the only element v in B^* for which

$$\langle \varphi_n, v \rangle = 0, \qquad \forall n \in \mathbb{Z},$$

is the zero element. Remembering that

$$\hat{\varphi}_n(t) = e^{-int}\hat{\varphi}(t), \qquad t \in \mathbb{R}, \quad n \in \mathbb{Z},$$

we obtain from the preceding observations that for every v in B^* and n in $\mathbb{Z}$,

$$\langle \varphi_n, v \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\varphi}(-t)\, \hat{v}(t)\, e^{int}\, dt = c_{-n}(P(\tilde{\hat{\varphi}}\hat{v})).$$

This implies the assertion, by the unicity theorem for Fourier series. ∎

In order to apply the proposition in concrete cases we need to introduce certain classes of subsets of $\mathbb{R}$, which will play a central role in the sequel.

In all that follows, measurable subsets of $\mathbb{R}$ will be defined modulo null sets (i.e., sets of measure zero). More precisely, we shall identify two measurable subsets of $\mathbb{R}$ whose symmetric difference is a null set. Accordingly, two measurable subsets whose intersection is a null set, will be considered disjoint. Similarly, if E is a measurable subset of $\mathbb{R}$, and f and g are measurable functions on $\mathbb{R}$ which are equal a.e. on E, we shall say that $f = g$ on E. A similar convention will be adopted for inequalities between functions.

The measure of a measurable subset E of $\mathbb{R}$ will be denoted by $|E|$.

For a subset A of $\mathbb{R}$ and a real number y we shall denote by $A + y$ the set $\{x + y, x \in A\}$.

DEFINITION 2. A measurable subset S of $\mathbb{R}$ will be said to be of special form it the sets $S + 2\pi n$, $n \in \mathbb{Z}$, are mutually disjoint.

This definition was introduced (in equivalent form) in [12]. In this reference, a measurable subset $S \subset \mathbb{R}$ is said to be of special form if

$$P\chi_S \leqslant 1,$$

where χ_S denotes as usual the characteristic function of S. It is clear that both definitions are equivalent.

As noted in [12], a measurable subset $S \subset \mathbb{R}$ is of special form if and only if

$$S = \bigcup_{n \in \mathbb{Z}} (S_n - 2\pi n),$$

where S_n, $n \in \mathbb{Z}$, are mutually disjoint measurable subsets of $[-\pi, \pi]$. It is readily verified that these subsets are defined by

$$S_n = (S + 2\pi n) \cap [-\pi, \pi], \qquad n \in \mathbb{Z}.$$

It follows from these observations that if S is of special form, then $|S| \leqslant 2\pi$, and that $|S| = 2\pi$ if and only if

$$\bigcup_{n \in \mathbb{Z}} (S + 2\pi n) = \mathbb{R},$$

or equivalently,

$$P\chi_S = 1.$$

The other classes of sets, are associated with vector spaces of measurable functions on $\mathbb{R}$.

Definition 3. If X is a vector space of measurable functions on $\mathbb{R}$, then a measurable subset $E \subset \mathbb{R}$ will be called a set of uniqueness for X if the only function in X that vanishes on E is the zero function.

Remark. A word of caution seems appropriate here. In view of our conventions concerning measurable subsets, the above definition means, of course, that E is a set of uniqueness for X, if f is X and vanishes a.e. on E, then f vanishes a.e. on $\mathbb{R}$. For example, if X is the vector space of all continuous functions on $\mathbb{R}$, or the vector space $C_0(\mathbb{R})$, and E is the set of rational numbers, then every function in X that vanishes on E is the zero function. However, E is not a set of uniqueness for X according to our definition, since it is a null set. It is clear that in these cases, the sets of uniqueness for X are precisely the measurable subsets of $\mathbb{R}$ whose intersections with every open interval have positive measure.

Before turning to the first application of Proposition 2.2, we introduce a notation and make an observation.

If X is a vector space of tempered distributions on $\mathbb{R}$, we shall denote by $\mathscr{F}X$ the vector space of all Fourier transforms of elements in X.

As observed before, if B is a Banach space of class $\mathscr{H}_k$, then B^* is included in $H^{2,-k}(\mathbb{R})$, so it is a vector space of tempered distributions, and $\mathscr{F}B^*$ is included in $L^2(\mathbb{R}, (1+x^2)^{-k}\,dx)$, hence it is a vector space of measurable functions on $\mathbb{R}$ (which are also locally integrable).

THEOREM 2.3. *If B is a Banach space of class $\mathscr{H}$, and the vector space $\mathscr{F}B^*$ has a set of uniqueness of special form, then B is singly generated. Moreover, for every N-function M, one can choose the generator to be an element of $H^{2,\infty}(\mathbb{R})$, which is the restriction to $\mathbb{R}$ of a function in $E_M(\mathbb{C})$.*

Proof. Let S be a set of uniqueness of special form for $\mathscr{F}B^*$, assume that M is an N-function, and denote by K its complementary function defined as usual (cf. [17, Ch. I]) by

$$K(t)=\sup_{s\geqslant 0}\,[st-M(s)], \qquad t\in[0,\infty),$$

and consider the function w on $\mathbb{R}$ defined by

$$w(t)=\exp(K(|t|)), \qquad t\in\mathbb{R}.$$

Let h be a measurable function on $\mathbb{R}$ such that

$$h(t)\neq 0, \qquad \forall t\in S,$$

and

$$hw\chi_S\in L^2(\mathbb{R}).$$

Consider the function $f=h\chi_S$, and set $\varphi=\hat{f}$. We claim that φ is a generator of B, which has all the required properties.

First observe that since K is also an N-function [17, Ch. I, Sect. 2], we have that

$$\lim_{t\to\infty}\frac{K(t)}{t}=\infty, \tag{$*$}$$

and therefore f is in $L^2(\mathbb{R},(1+x^2)^j\,dx)$ for every positive integer j, and this implies that the function φ is in $H^{2,\infty}(\mathbb{R})$. To show that it is a generator of B, we apply Proposition 2.2.

Assume that v is in B^* and $P(\tilde{\hat{\varphi}}\hat{v})=0$. Noting that $\tilde{\hat{\varphi}}=2\pi f$ (by the inversion theorem for the Fourier transform on $L^2(\mathbb{R})$), we obtain that

$$2\pi\sum_{j=-\infty}^{\infty} f_{2\pi j}\hat{v}_{2\pi j}=P(\tilde{\hat{\varphi}}\hat{v})=0,$$

and therefore, since S is of special form, and

$$f_{2\pi j}(x)=\chi_S(x-2\pi j)\,h(x-2\pi j), \qquad x\in\mathbb{R}, \quad j\in\mathbb{Z},$$

we get that all the terms in this sum vanish, hence in particular

$$\hat{v} h \chi_S = 0.$$

Thus remembering that $h \neq 0$ on S and that S is a set of uniqueness for $\mathscr{F} B^*$, we conclude that $\hat{v} = 0$, and consequently also $v = 0$. Thus, by Proposition 2.2, φ is a generator for B.

To prove the remaining assertion, observe that the fact that S has finite measure implies that fw is in $L^1(\mathbb{R})$, and therefore using again condition $(*)$, we see that the function F on $\mathbb{C}$ defined by

$$F(z) = \int_{-\infty}^{\infty} f(t)\, e^{-izt}\, dt, \qquad z \in \mathbb{C},$$

is entire. Using Young's inequality [17, p. 12], we obtain that

$$|F(z)| \leqslant \|fw\|_{L^1(\mathbb{R})} \exp(M(|z|)), \qquad \forall z \in \mathbb{C}.$$

This shows that the function F is in $E_M(\mathbb{C})$, and since its restriction to $\mathbb{R}$ is φ, the proof is complete. ∎

Next, we show that the hypothesis of Theorem 2.3 also implies that the generator can be chosen so that its non-negative integer translates are already complete.

THEOREM 2.4. *The function φ in Theorem 2.3, can be chosen so that its non-negative integer translates $\{\varphi_n, n = 0, 1, \ldots\}$ are already complete in B.*

Proof. We keep all the notations in the proof of Theorem 2.3 and impose on the function h the additional condition

$$\int_S \log |h(t)|\, dt = -\infty.$$

We claim that this implies that the non-negative integer translates of φ are complete in B. To show this, let $v \in B^*$, consider the function

$$g = P(\hat{v} h \chi_S),$$

and assume that

$$c_n(g) = 0, \qquad n = 0, 1, \ldots.$$

We shall show that this assumption implies that $g = 0$, and this will imply the assertion by the proofs of Proposition 2.2 and Theorem 2.3.

It follows from the assumption above that the function $\bar{g}$ (the complex conjugate of g) belongs to the Hardy space $H^1(\mathbb{T})$, and therefore by a classical theorem of F. Riesz (see [16, p. 90]), the desired conclusion will follow if we show that

$$\log |g| \notin L^1(\mathbb{T}).$$

To this end, consider the sets

$$S_n = (S + 2\pi n) \cap [-\pi, \pi], \qquad n \in \mathbb{Z},$$

and

$$A = [-\pi, \pi] \Big\backslash \Big(\bigcup_{n \in \mathbb{Z}} S_n \Big).$$

Since S is of special form, the sets $S_n, n \in \mathbb{Z}$, are mutually disjoint; thus using the 2π-periodicity of the function g, we obtain that

$$\begin{aligned} \int_{\pi}^{\pi} \log |g(t)|\, dt &= \int_A \log |g(t)|\, dt + \sum_{n=-\infty}^{\infty} \int_{S_n} \log |g(t)|\, dt \\ &= \int_A \log |g(t)|\, dt + \int_S \log |g(t)|\, dt. \end{aligned}$$

Therefore, observing that $g = h\hat{v}$ on S, we get that

$$\int_{-\pi}^{\pi} \log |g(t)|\, dt \leqslant \int_{-\pi}^{\pi} |g(t)|\, dt + \int_S |\hat{v}(t)|\, dt + \int_S \log |h(t)|\, dt = -\infty.$$

The last equality follows from the assumption on h, since g is in $L^1(\mathbb{T})$, $\hat{v}$ is locally integrable, and S has finite measure. This concludes the proof. ▮

Remarks. 1. The preceding proof shows that if $|S| < 2\pi$, then the conclusion of the theorem follows without any additional assumption on h. Indeed, in this case, $|A| > 0$, and since it is easily seen that $g = 0$ on A, we obtain that $\log |g|$ is not in $L^1(\mathbb{T})$.

2. The proof of the theorem also shows that if S is a set of uniqueness of special form for $\mathscr{F}B^*$, then the set $E = A \cup S$ also has these properties (since $E = \bigcup_{n \in \mathbb{Z}} (S_n^* - 2\pi n)$, where $S_0^* = S_0 \cup A$, and $S_n^* = S_n$ for $n \neq 0$), and $|E| = 2\pi$.

We shall prove in the next section that if B is one of the Banach spaces listed in the statement of Theorem 2.1, then there exists a set of uniqueness of special form S for $\mathscr{F}B^*$, such that the sequence

$$|S \cap [2\pi n, 2\pi(n+1)]|, \qquad n \in \mathbb{Z},$$

tends to zero arbitrary fast as $|n| \to \infty$. We show that this fact implies that B has a generator which satisfies all the conditions of Theorem 2.3 and, in addition, its translates form an orthonormal system in $L^2(\mathbb{R})$.

THEOREM 2.5. *Assume that B is a Banach space of class $\mathscr{H}$ and that for every sequence of positive numbers $\{c_n\}_{n \in \mathbb{N}}$, there exists a set of uniqueness of special form S for $\mathscr{F}B^*$, such that*

$$|S \cap [2\pi n, 2\pi(n+1)]| \leqslant c_{|n|}, \qquad \text{for} \quad |n| \geqslant 1, \quad n \in \mathbb{Z}.$$

Then one can choose the function φ in Theorem 2.3, so that it satisfies all the conditions listed there and, in addition, its integer translates form an orthonormal system in $L^2(\mathbb{R})$.

Proof. The hypothesis above implies that for every continuous function w on $\mathbb{R}$, there exists a set of uniqueness of special form S for $\mathscr{F}B^*$, such that

$$w\chi_S \in L^2(\mathbb{R}).$$

Examining the proof of Theorem 2.3, we see that this implies that all its conclusions can be fulfilled by choosing the function φ as $(2\pi)^{-1}\hat{\chi}_S$, for an appropriate choice of S. By the second remark following the proof of Theorem 2.4, we may also assume that $|S| = 2\pi$. Thus noting that $\hat{\varphi} = \tilde{\chi}_S$, we get from the observation following Definition 2 that

$$P(|\hat{\varphi}|^2) = 1.$$

It is known (cf. [11] or [19, Ch. II]), and follows from the Poisson summation formula, that this condition is equivalent to the orthonormality of the integer translates of φ in $L^2(\mathbb{R})$, and the proof is complete. ▌

We now turn to the construction of a generator whose integer translates form a minimal system. We begin with some notations and preliminary observations.

In what follows we shall denote for every real number s, by T_s the translation operator acting on functions f on $\mathbb{R}$ by

$$T_s f = f_s.$$

We denote by $\mathscr{S}$ the Schwartz space of rapidly decreasing C^∞ functions on $\mathbb{R}$ and by $\mathscr{S}'$ its dual space, that is, the space tempered distributions on $\mathbb{R}$. We recall that for a real number s, the translate by s of a tempered distribution v, is the tempered distribution v_s defined by

$$\langle u, v_s \rangle = \langle T_{-s} u, v \rangle, \qquad u \in \mathscr{S}.$$

We shall denote the transformation on $\mathscr{S}'$,

$$v \to v_s, \qquad v \in \mathscr{S}',$$

also by T_s.

As observed before, if B is a Banach space of class $\mathscr{H}$, then $B^* \subset \mathscr{S}'$. Therefore, if B is mapped continuously into itself by the transformations T_n, $n \in \mathbb{Z}$, the same is true for B^*, since the adjoint of the operator T_n on B is the operator T_{-n} on B^*. We shall use these observations in the proof below.

THEOREM 2.6. *Assume that B is a Banach space of class $\mathscr{H}$ that satisfies the hypothesis of Theorem 2.5 and is mapped continuously into itself by the transformations T_n, $n \in \mathbb{Z}$. If B^* contains an element ψ such that $\hat{\psi}$ is a continuous function on $\mathbb{R}$ which is nowhere zero, then B has a generator that satisfies all the conditions stated in Theorem 2.1, and in addition, its integer translates form a minimal system.*

Proof. Let ψ be an element of B^* with the properties described above, and consider the function

$$h = (2\pi\hat{\psi})^{-1}.$$

Then h is a continuous function on $\mathbb{R}$, and therefore since B satisfies the hypothesis of Theorem 2.5, we get as in its proof that for every continuous function w on $\mathbb{R}$, there exists a set of uniqueness of special form S for $\mathscr{F}B^*$, of measure 2π, such that

$$hw\chi_S \in L^2(\mathbb{R}).$$

Hence by the arguments in the proof of Theorem 2.3, we obtain that with an appropriate choice of S, the function $\varphi = \widehat{h\chi_S}$ will satisfy all the conclusions of that theorem.

We shall now show that $\{\varphi_n, n \in \mathbb{Z}\}$ is a minimal system. For this, we have to prove that this sequence has a biorthogonal sequence in B^*. We claim that the sequence $\{\psi_n, n \in \mathbb{Z}\}$ of integer translates of ψ has this property. Indeed, it follows from the definition of φ and the Fourier inversion formula that

$$\tilde{\hat{\varphi}}\hat{\psi} = \chi_S$$

and therefore for all j and k in $\mathbb{Z}$,

$$\langle \varphi_j, \psi_k \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(j-k)t} \hat{\varphi}(-t)\, \hat{\psi}(t) = c_{j-k}(P\chi_S) = \delta_{j,k}.$$

The last equality follows from the fact that the assumption that $|S| = 2\pi$ implies that

$$P\chi_S = 1.$$

This completes the proof. ▮

Remarks. 1. As already observed before, all the Banach spaces listed in the statement of Theorem 2.1 satisfy the hypothesis of Theorem 2.5. Since their dual spaces include the space $\mathscr{S}$, they also satisfy the hypothesis of Theorem 2.6 (for example with $\psi(x) = e^{-x^2}$) and therefore also its conclusion.

2. Each of the Banach spaces $C_0^k(\mathbb{R})$, $k = 0, 1, ...,$ and $H^{p,k}(\mathbb{R})$, $2 < p < \infty$, $k = 1, 2, ...,$ has a generator which satisfies conditions 2 and 3 of Theorem 2.1 simultaneously. This follows from the fact that the evaluation functional

$$\delta_0 : f \to f(0)$$

belongs to their duals. (This is clear for the first class of spaces, and for the second one, it follows from the proof of the Sobolev embedding theorem.) Therefore since $\hat{\delta}_0 = 1$, the hypothesis of Theorem 2.6 is satisfied with $\psi = \delta_0$. Thus from the proof of that theorem we obtain that for these spaces, one can choose the function φ to be $(2\pi)^{-1}\hat{\chi}_S$, for an appropriate choice of S, and consequently by the proof of Theorem 2.5, its integer translates will also from an orthonormal system in $L^2(\mathbb{R})$.

3. Conditions 1 and 3 in Theorem 2.1 cannot be satisfied simultaneously. In fact, if B is any Banach space of tempered distributions on $\mathbb{R}$, which is mapped continuously into itself by the transformations T_n, $n \in \mathbb{Z}$, and u is an element of B such that the sequence $\{u_n, n = 0, 1, \ldots\}$ is complete, then it is not minimal (hence the same is true also for the sequence $\{u_n, n \in \mathbb{Z}\}$). This follows from the fact that if this sequence is complete, then its closed linear span contains in particular the element u_{-1}, and therefore since

$$T_1 u_n = u_{n+1}, \qquad \forall n \in \mathbb{Z},$$

the element $u = u_0$ is contained in the closed linear span of the sequence $\{u_n, n \in \mathbb{N}\}$.

In concluding this section it is worth nothing that while the existence of singly generated spaces among the classical Banach spaces of functions on $\mathbb{R}$ is rather exotic, if one considers all translates, and not just integer ones, this is a common fact and holds for most of the classical spaces, in particular for the spaces $L^p(\mathbb{R})$, $1 \leqslant p < \infty$, and the corresponding Sobolev spaces. In fact, every Banach space B of tempered distributions on $\mathbb{R}$ which contains the Fréchet–Schwartz space $\mathscr{S}$, and the imbedding is dense and continuous, contains a function φ such that the translates $\{\varphi_t, t \in \mathbb{R}\}$ are complete. Moreover, every function in φ whose Fourier transform has no zeros (for example, $\varphi(x) = e^{-x^2}$) has this property. This can be seen as follows. If φ is a function in $\mathscr{S}$ such that $\hat{\varphi}$ has no zeros, then it follows from the results in [24, Ch. VII, Sect. 10] that the only element v in $\mathscr{S}'$ such that

$$\varphi * v = 0$$

is the zero element. Hence by the Hahn–Banach theorem, the linear span of the set $\{\varphi_t, t \in \mathbb{R}\}$ is dense in the Fréchet space $\mathscr{S}$, and since this space is continuously and densely imbedded in B, it is also dense in B.

However, not every separable translation invariant Banach space of functions on $\mathbb{R}$ is the closed span of translates of a single function. As shown in [1], there exist closed translation invariant subspaces of $L^1(\mathbb{R})$, which are not even the closed span of translates of finitely many functions.

3. SETS OF UNIQUENESS

In view of the results of the preceding section, Theorem 2.1 will be proved if we show that if B is one of the Banach spaces listed in its statement, then the vector space $\mathscr{F}B^*$ has a set of uniqueness of special form, which satisfies the hypothesis of Theorem 2.5. This section is devoted to the proof of this fact. We begin with some notations.

For every real number t and every function u in $\mathscr{S}$, we shall denote by M_t and C_u the operators on $\mathscr{S}'$ of multiplication by the function

$$x \to e^{itx}, \qquad x \in \mathbb{R},$$

and convolution by the function u, respectively.

It is clear that the vector space $\mathscr{S}'$ is invariant under the operators M_t, $t \in \mathbb{R}$, and it is well known (cf. [24, p. 246]) that it is also invariant under the convolution operators C_u, $u \in \mathscr{S}$.

We denote as usual by $L^1_{\text{loc}}(\mathbb{R})$ the vector space of locally integrable functions on $\mathbb{R}$. This is a Fréchet space, with respect to the sequence of seminorms $\{p_n\}_{n \in \mathbb{N}}$ defined by

$$p_n(f) = \int_{-n}^{n} |f(t)| \, dt, \qquad f \in L^1_{\text{loc}}(\mathbb{R}).$$

Definition. A vector space X, which is included in $\mathscr{S}'$, will be called admissible if it satisfies the following conditions:

(1) $\mathscr{F}X \subset L^1_{\mathrm{loc}}(\mathbb{R})$.

(2) X is invariant under the operators M_t, $t \in \mathbb{R}$.

(3) X is invariant under the operators C_u, $u \in \mathscr{S}$.

Note that if X is an admissible vector space, then the vector space $\mathscr{F}X$ is translation invariant by condition 2 and is invariant under multiplication by functions in $\mathscr{S}$ by condition 3.

If X is a vector space included in $\mathscr{S}'$ and τ is a positive number, we shall denote by X_τ, the vector space of all elements in X whose Fourier transforms are supported by the interval $[-\tau, \tau]$.

In view of the Paley–Wiener–Schwartz theorem, X_τ consists of all elements in X, which are the restriction to $\mathbb{R}$ of an entire function of exponential type τ. If X is a Banach space which is continuously imbedded in $\mathscr{S}'$ (with respect to the ω^* topology of that space), then it is easily verified that for every $\tau > 0$, the subspace X_τ is closed, and hence is also a Banach space.

The relevance of these notions to our objective comes from the following.

Proposition 3.1. *If X is an admissible vector space and $\{S_n\}_{n \in \mathbb{Z}}$ is a sequence of mutually disjoint measurable subsets of $[-\pi, \pi]$, which are sets of uniqueness for the vector space $\mathscr{F}(X_\pi)$, then the set*

$$S = \bigcup_{n \in \mathbb{Z}} (S_n - 2\pi n)$$

is a set of uniqueness of special form for the vector space $\mathscr{F}X$.

Proof. It is clear that S is of special form. To show that it is a set of uniqueness for $\mathscr{F}X$, assume that g is a function in that space which vanishes on S. Then for every function u in $\mathscr{S}$ with support in $[-\pi, \pi]$ and every integer n, the function $ug_{2\pi n}$ is in $\mathscr{F}(X_\pi)$ and vanishes on S_n, and since this is a set of uniqueness for $\mathscr{F}(X_\pi)$, we get that

$$ug_{2\pi n} = 0, \qquad \forall n \in \mathbb{Z}.$$

Since u is an arbitrary function in $\mathscr{S}$ with support in $[-\pi, \pi]$, this implies that $g_{2\pi n}$ vanishes on the interval $[-\pi, \pi]$, for every integer n, and therefore $g = 0$. This completes the proof. ∎

Before applying the proposition to the Banach spaces listed in the statement of our main theorem, we make some observations.

Assume that B is a Banach space of class $\mathscr{H}$ which is included in $\mathscr{S}'$ and is mapped continuously into itself by the operators M_t and C_u, for every $t \in \mathbb{R}$ and $u \in \mathscr{S}$. Then the dual space B^* is admissible. Condition 1 holds, since as observed before, B^* is always included in $\mathscr{S}'$, and $\mathscr{F}B^*$ is included in $L^1_{\mathrm{loc}}(\mathbb{R})$. Conditions 2 and 3 hold since the adjoints of the operators M_t and C_u on B are the operators M_t and $C_{\tilde{u}}$ on B^*.

It follows from these observations that the dual space of each of the Banach spaces described in Theorem 2.1 is admissible. It is clear that all these spaces are included in $\mathscr{S}'$ and are mapped continuously into themselves by the operators M_t, $t \in \mathbb{R}$, and it is well known that the spaces $C_0^k(\mathbb{R})$ and $H^{p,k}(\mathbb{R})$ are also mapped continuously into themselves by the operators C_u, $u \in \mathscr{S}$.

That the same is true for the spaces $L^2(\mathbb{R}, \rho(x)\,dx)$, where ρ is a tempered weight function, follows from [14, Th. 2.2.4], since in the notation there, $\mathscr{F}L^2(\mathbb{R}, \rho(x)\,dx) = \mathfrak{B}_{2,\sqrt{\rho}}$.

Thus by the observation in the beginning of this section and Proposition 3.1, Theorem 2.1 will follow from:

THEOREM 3.2. *If B is one of the Banach spaces described in the statement of Theorem* 2.1, *then for every sequence of positive numbers $\{a_n\}_{n\in\mathbb{N}}$, there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ of mutually disjoint measurable subsets of $[-\pi, \pi]$, which are sets of uniqueness for the vector space $\mathscr{F}(B^*_\pi)$, and*

$$|E_n| \leqslant a_n, \qquad \forall n \in \mathbb{N}.$$

We begin with the proof for the spaces $C_0^k(\mathbb{R})$, $k = 0, 1, \ldots$. The dual space of $C_0(\mathbb{R})$ is $M(\mathbb{R})$, the space of all bounded complex Borel measures on $\mathbb{R}$, and for $k = 1, 2, \ldots$, the dual space of $C_0^k(\mathbb{R})$ is the vector space $M^{-k}(\mathbb{R})$, which consists of all linear combinations of (distributional) derivatives of order at most k, of elements in $M(\mathbb{R})$. Since the elements of $\mathscr{F}M(\mathbb{R})$ are continuous functions on $\mathbb{R}$, the same is true for the elements of $\mathscr{F}M^{-k}(\mathbb{R})$, $k = 1, 2, \ldots$. Therefore every measurable subset of $[-\pi, \pi]$ which has intersection of positive measure with every open interval included in $[-\pi, \pi]$ is a set of uniqueness for each of the vector spaces $\mathscr{F}(M(\mathbb{R})_\pi)$ and $\mathscr{F}(M^{-k}(\mathbb{R})_\pi)$, $k = 1, 2, \ldots$. Thus the claim of Theorem 2.1 for the spaces $C_0^k(\mathbb{R})$, $k = 0, 1, \ldots$, is a consequence of the following:

PROPOSITION 3.3. *For every sequence of positive numbers $\{a_n\}_{n\in\mathbb{N}}$, there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ of mutually disjoint measurable subsets of $[-\pi, \pi]$, such that each of them has intersection of positive measure with every open interval included in $[-\pi, \pi]$, and*

$$|E_n| \leqslant a_n, \qquad \forall n \in \mathbb{N}.$$

This result is known and elementary. It also follows from Theorem 3.6 to be proved in the sequel (see the remark at the end of this section).

However, it seems to be of interest to have a direct proof that is independent of the more elaborate arguments involved in the proof of that theorem. Since we were unable to find such a proof in the literature, we present one here. It is based on a Baire category argument. One can also give a direct proof, but it is somewhat longer.

Proof. Let $\mathfrak{M}$ denote the collection of all measurable subsets of $[-\pi, \pi]$. As usual, we identify two elements A of B of $\mathfrak{M}$, whose symmetric difference

$$A \triangle B = (A \backslash B) \cup (B \backslash A)$$

is a null set. It is well known that with the metric defined by

$$d(A, B) = |A \triangle B|, \qquad A, B \in \mathfrak{M},$$

$(\mathfrak{M}, d)$ is a complete metric space.

Let $\mathfrak{M}_\infty$ be the Cartesian product of countably infinite many copies of $\mathfrak{M}$. With the metric

$$d_\infty(x, y) = \sum_{j=1}^{\infty} 2^{-j} d(A_j, B_j), \quad x = (A_j)_{j \in \mathbb{N}}, \quad y = (B_j)_{j \in \mathbb{N}} \in \mathfrak{M}_\infty,$$

$(\mathfrak{M}_\infty, d_\infty)$ is also a complete metric space.

Let $\{a_j\}_{j \in \mathbb{N}}$ be a sequence of positive numbers, and consider the subset of $\mathfrak{M}_\infty$,

$$W = \{(A_j)_{j \in \mathbb{N}} \in \mathfrak{M}_\infty : |A_j \cap A_k| = 0, \text{ for } j \neq k, \text{ and } |A_j| \leqslant a_j, \forall j \in \mathbb{N}\}.$$

It is readily verified that W is closed. Let $\{I_n, n \in \mathbb{N}\}$ be the set of all open intervals included in $[-\pi, \pi]$, with rational end points. We have to show that there exists an element $(E_j)_{j \in \mathbb{N}}$ in W, such that

$$|E_j \cap I_n| > 0, \qquad \forall j, n \in \mathbb{N}.$$

To this end, consider for every pair of positive integers k and n the subset of W,

$$W_{k,n} = \{(F_j)_{j \in \mathbb{N}} \in W : |F_k \cap I_n| = 0\}.$$

These are closed subsets of W, and the claim of the theorem is equivalent to the assertion that their union is not equal to W. This will follow from the Baire category theorem, if we show that each of these sets has empty interior in the relative topology of W. For this, fix a pair of positive integers k and n, an element $x_0 = (C_j)_{j \in \mathbb{N}}$ in W, a positive number δ, and consider the set

$$U(x_0, \delta) = \{x \in W: d_\infty(x_0, x) < \delta\}.$$

In order to show that this set is not included in $W_{k,n}$, we choose an open interval $J \subset I_n$ as follows:

If $|C_k| > 0$, consider a measurable subset $C \subset C_k$, such that $0 < |C| < \delta$, and require that $|J| \leqslant |C|$.

If $|C_k| = 0$, require that $|J| \leqslant \min\{a_k, \delta\}$. Let $y = (D_j)_{i \in \mathbb{N}}$ be the element of W defined by

$$\begin{aligned} D_j &= C_j \backslash J, && \text{if} \quad j \neq k, \\ D_k &= (C_k \backslash C) \cup J, && \text{if} \quad |C_k| > 0, \\ D_k &= J, && \text{if} \quad |C_k| = 0. \end{aligned}$$

It is easy to check that

$$y \in U(x_0, \delta) \backslash W_{k,n}.$$

Thus $W_{k,n}$ has empty interior, and the proposition is proved. ∎

The proof of Theorem 3.2 for the other Banach space B listed in the statement of Theorem 2.1 requires a closer examination of the corresponding spaces B^*_π.

We observe first that for every $1 < p < \infty$ and every non-negative integer k,

$$H^{p,k}(\mathbb{R})^*_\pi = L^q(\mathbb{R})_\pi, \qquad \text{with} \quad q = p(p-1)^{-1}.$$

To see this, recall (see [25]) that

$$H^{p,k}(\mathbb{R})^* = H^{q,-k}(\mathbb{R}), \qquad \text{with} \quad q = p(p-1)^{-1},$$

where $H^{q,-k}(\mathbb{R})$ consists of all linear combinations of distributional derivatives of order at most k, of functions in $L^q(\mathbb{R})$. Since for every u in $\mathscr{S}$ and f in $L^q(\mathbb{R})$,

$$u * f^{(j)} = u^{(j)} * f, \qquad j = 0, 1, ...,$$

and $L^q(\mathbb{R})$ is invariant under the convolution operators C_u, $u \in \mathscr{S}$, we see that these operators map the vector space $H^{q,-k}(\mathbb{R})$ into $L^q(\mathbb{R})$. Therefore, since for every g in $\mathscr{S}$ such that $\hat{g} = 1$ on $[-2\pi, 2\pi]$,

$$g * v = v, \qquad \forall v \in H^{q,-k}(\mathbb{R})_\pi$$

(see [24, Ch. V, Th. 2, and Ch. VI, Sect. 10]), we get that

$$H^{q,-k}(\mathbb{R})_\pi \subset L^q(\mathbb{R})_\pi.$$

Since the inclusion in the other direction is clear, the equality is established.

Consequently, it remains to prove Theorem 3.2 for the spaces $L^p(\mathbb{R})$, $2 < p < \infty$, and $L^2(\mathbb{R}, \rho(x)\,dx)$, where ρ is a tempered weight function in $C_0(\mathbb{R})$. For this, we require the following result, which is an extension of a result of Pólya and Plancherel [22, p. 126] (see also [5, Th. 6.7.15], and [9]).

THEOREM 3.4. *If $1 \leqslant q < \infty$, and γ is a tempered weight function on $\mathbb{R}$, then there exists a positive constant c (which depends only on q on γ) such that*

$$\sum_{n=-\infty}^{\infty} |f(n)|^q\,\gamma(n) \leqslant c \int_{-\infty}^{\infty} |f(x)|^q\,\gamma(x)\,dx, \qquad \forall f \in L^q(\mathbb{R}, \gamma(x)\,dx)_\pi.$$

Proof. Let g be a function in $\mathscr{S}$ such that $\hat{g} = 1$ on $[-2\pi, 2\pi]$, then

$$f = f * g, \qquad \forall f \in L^q(\mathbb{R}, \gamma(x)\,dx)_\pi,$$

that is,

$$f(x) = \int_{-\infty}^{\infty} f(y)\,g(x-y)\,dy, \qquad x \in \mathbb{R},\ f \in L^q(\mathbb{R}, \gamma(x)\,dx)_\pi.$$

Consider the function β on $\mathbb{R}$ defined by

$$\beta(x) = (\gamma(x))^{1/q}, \qquad x \in \mathbb{R}.$$

Since γ is a tempered weight function, the same is true for β, hence there exist positive numbers c_1 and r such that

$$\beta(x) \leqslant c_1 \beta(y)(1 + |x-y|)^r, \qquad x, y \in \mathbb{R}.$$

Since g is in $\mathscr{S}$, there exists a constant $c_2 > 0$ such that

$$|g(x)| \leqslant c_2(2 + |x|)^{-r-2}, \qquad x \in \mathbb{R},$$

and therefore, we obtain from the formula above that there exists a constant $c_3 > 0$ such that for every function f in $L^q(\mathbb{R}, \gamma(x)\,dx)_\pi$,

$$|f(x)\,\beta(x)| \leqslant c_3 \int_{-\infty}^{\infty} |f(y)\,\beta(y)|\,(2 + |x-y|)^{-2}\,dy, \qquad x \in \mathbb{R}.$$

Thus, denoting by h the function on $\mathbb{R}$ defined by

$$h(x)=(1+|x|)^{-2}, \qquad x\in\mathbb{R},$$

and using the fact that

$$(2+|x|)^{-2}\leqslant h(x+t), \qquad x\in\mathbb{R}, \quad 0\leqslant t\leqslant 1,$$

we get that for every such function f,

$$|f(x)\,\beta(x)|\leqslant c_3\,|f\beta| * h(x+t), \qquad x\in\mathbb{R}, \quad 0\leqslant t\leqslant 1,$$

and therefore

$$\sum_{n=-\infty}^{\infty} |f(n)|^q\,\gamma(n)\leqslant c_3^q \sum_{n=-\infty}^{\infty} \int_0^1 [\,|f\beta| * h(n+t)]^q\,dt$$
$$= c_3^q\,\|\,|f\beta| * h\|_{L^q(\mathbb{R})}^q \leqslant c_3^q\,\|h\|_{L^1(\mathbb{R})}^q\,\|f\beta\|_{L^q(\mathbb{R})}^q.$$

The last inequality follows from Young's inequality for convolutions. This completes the proof. ∎

Remarks. 1. For $\gamma\equiv 1$ this result is due to Pólya and Plancherel [22]. Our proof is different from and considerably shorter than theirs. For our purpose, the general case is needed only for $q=2$, and the case $\gamma\equiv 1$ only for $2<q<\infty$. We also note that by the proof above, the result remains true if π is replaced by any positive number τ (the constant c will then depend also on τ). This is also the case in [22], where the result is actually stated for general τ.

2. It is also proved in [22] that for the spaces $L^q(\mathbb{R})_\pi$, $1<q<\infty$, there is also a corresponding inequality in the other direction, namely, there exists a positive constant b (which depends only on q) such that for every function f in $L^q(\mathbb{R})_\pi$,

$$\int_{-\infty}^{\infty} |f(x)|^q\,dx\leqslant b\sum_{n=-\infty}^{\infty} |f(n)|^q.$$

In order to apply the conclusion of Theorem 3.4 to the proof of the remaining part of Theorem 3.2, we have to introduce several Banach spaces of functions on $\mathbb{T}$. We observe first that, if $1<q<2$, then by the Hausdorff–Young theorem, $\mathscr{F}L^q(\mathbb{R})$ is included in $L^p(\mathbb{R})$, with $p=q(q-1)^{-1}$, and therefore the vector space $\mathscr{F}(L^q(\mathbb{R})_\pi)$ is included in $L^1(\mathbb{R})$; so it makes sense to consider the 2π-periodizations of its elements. The same is true for the spaces $L^2(\mathbb{R}, \gamma(x)\,dx)$, where γ is a continuous function on $\mathbb{R}$ which is bounded below by a positive constant, since these spaces are included in $L^2(\mathbb{R})$. We use these facts below.

For $1<q<2$, we shall denote by $B_q(\mathbb{T})$ the vector space of all 2π-periodizations of functions in $\mathscr{F}(L^q(\mathbb{R})_\pi)$. This is a Banach space with respect to the norm

$$\|P\hat{f}\|_{B_q(\mathbb{T})}=\|f\|_{L^q(\mathbb{R})}, \qquad f\in L^1(\mathbb{R})_\pi.$$

For a tempered weight function γ on $\mathbb{R}$ such that

$$\inf_{x\in\mathbb{R}} \gamma(x)>0,$$

we shall denote by $B_2(\mathbb{T},\gamma)$ the vector space of all 2π-periodizations of functions in $\mathscr{F}L^2(\mathbb{R},\gamma(x)\,dx)_\pi$. This is also a Banach space with norm defined by

$$\|P\hat{f}\|_{B_2(\mathbb{T},\gamma)}=\|f\|_{L^2(\mathbb{R},\gamma(x)\,dx)}, \qquad f\in L^2(\mathbb{R},\gamma(x)\,dx)_\pi.$$

For $1<q<2$, we denote by $A_q(\mathbb{T})$ the Banach space of all functions g in $L^1(\mathbb{T})$, for which the norm

$$\|g\|_{A_q(\mathbb{T})}=\left(\sum_{n=-\infty}^{\infty}|c_n(g)|^q\right)^{1/q}$$

is finite. Finally, for a sequence of positive numbers $\omega=\{\omega(n)\}_{n\in\mathbb{Z}}$ such that

$$\inf_{n\in\mathbb{Z}} \omega(n)>0,$$

we shall denote by $A_2(\mathbb{T},\omega)$ the Banach space of all functions g in $L^2(\mathbb{T})$, for which the norm

$$\|g\|_{A_2(\mathbb{T},\omega)}=\left(\sum_{n=-\infty}^{\infty}|c_n(g)|^2\,\omega(n)\right)^{1/2}$$

is finite.

Observe that if v is a tempered distribution such that $\hat{v}$ is in $L^1(\mathbb{R})$, then v is in $C_0(\mathbb{R})$, and

$$c_n(P\hat{v})=(2\pi)^{-1}\,v(-n), \qquad \forall n\in\mathbb{Z}.$$

Applying this fact to the elements of $L^q(\mathbb{R})_\pi$ and $L^2(\mathbb{R},\gamma(x)\,dx)_\pi$, with q and γ as above, we see that an immediate consequence of Theorem 3.4 is:

COROLLARY 3.5. (a) *If* $1 < q < 2$, *then*

$$B_q(\mathbb{T}) \subset A_q(\mathbb{T}).$$

(b) *If* γ *is a tempered weight function on* $\mathbb{R}$, *which is bounded below by a positive constant, and* $\omega = \{\omega(n)\}_{n \in \mathbb{Z}}$ *is the sequence defined by*

$$\omega(n) = \gamma(-n), \qquad n \in \mathbb{Z},$$

then

$$B_2(\mathbb{T}, \gamma) \subset A_2(\mathbb{T}, \omega).$$

Remark. It follows from the second remark after the proof of Theorem 3.4 that the inclusion in the first part of the corollary is actually an equality.

It is clear that a measurable subset of $[-\pi, \pi]$ is a set of uniqueness for the space $\mathscr{F}(L^q(\mathbb{R}_\pi)$ for some $1 < q < 2$, if and only if it is a set of uniqueness for the space $B_q(\mathbb{T})$, and the same holds for the spaces $\mathscr{F}(L^2(\mathbb{R}, \gamma(x)\,dx)_\pi$ and $B_2(\mathbb{T}, \gamma)$, where γ is a weight function which satisfies the hypothesis in part (b) of the corollary.

If ρ is a tempered weight function in $C_0(\mathbb{R})$, then the function $\gamma = 1/\rho$ is a tempered weight function, such that

$$\lim_{x \to \pm\infty} \gamma(x) = \infty,$$

and

$$L^2(\mathbb{R}, \rho(x)\,dx)^* = L^2(\mathbb{R}, \gamma(x)\,dx),$$

with the duality implemented by the pairing

$$\langle u, v \rangle = \int_{-\infty}^{\infty} u(t)\, v(t)\, dt,$$

where $u \in L^2(\mathbb{R}, \rho(x)\,dx)$, and $v \in L^2(\mathbb{R}, \gamma(x)\,dx)$.

Observe that this pairing is compatible with the pairing described in Section 2, between a function in $H^{2,k}(\mathbb{R})$ and an element in the dual of a Banach space of class $\mathscr{H}_k$ (see also [14, Th. 2.2.9]).

Since γ is a tempered weight function, there exist positive constants c_1 and c_2 such that

$$c_1 \gamma(x) \leqslant \gamma(x+1) \leqslant c_2 \gamma(x), \qquad x \in \mathbb{R},$$

and therefore by Corollary 3.5 and the preceding observations, the conclusion of Theorem 3.2 for the spaces $L^p(\mathbb{R})$, $2 < p < \infty$, and $L^2(\mathbb{R}, \rho(x)\,dx)$, where ρ is a tempered weight function in $C_0(\mathbb{R})$, will follow from:

THEOREM 3.6. *Assume that* $1<q<2$ *and that* $\{\omega(n)\}_{n\in\mathbb{Z}}$ *is a sequence of positive numbers such that*

$$\lim_{n\to\pm\infty}\omega(n)=\infty,$$

and for some positive constants c_1 *and* c_2

$$c_1\omega(n)\leqslant\omega(n+1)\leqslant c_2\omega(n),\qquad n\in\mathbb{Z}.$$

If Y *is one of the Banach spaces* $A_q(\mathbb{T})$ *or* $A_2(\mathbb{T},\omega)$, *then for every sequence of positive numbers* $\{a_j\}_{j\in\mathbb{N}}$, *there exists a sequence* $\{E_j\}_{j\in\mathbb{N}}$ *of mutually disjoint measurable subsets of* $[-\pi,\pi]$, *which are sets of uniqueness for* Y, *and*

$$|E_j|\leqslant|a_j|,\qquad\forall j\in\mathbb{N}.$$

The existence of sets of uniqueness of arbitrary small measure for the spaces $A_q(\mathbb{T})$, $1<q<2$, was established in 1964, independently, by Y. Katznelson [15] (see also [16, p. 101] and D. J. Newman [20, Th. 5′]). Extensions of their result to a more general setting are given in [10], and related results appear in [21].

Remark. It should be noted here, that in the harmonic analysis literature (in particular in [15, 20, 10]) a measurable subset E of $[-\pi,\pi]$ is called a set of uniqueness for $A_q(\mathbb{T})$ if the set $[-\pi,\pi]\backslash E$ is a set of uniquenesss in our sense.

In order to prove the theorem simultaneously for the spaces $A_q(\mathbb{T})$ and $A_2(\mathbb{T},\omega)$, it is convenient to introduce a certain class of Banach spaces of functions on $\mathbb{T}$. We first make an observation.

If Y is a Banach space of functions on $\mathbb{T}$ which is included in $L^2(\mathbb{T})$, and the embedding is continuous, then every function f in $L^2(\mathbb{T})$ defines a bounded linear functional on Y, by

$$g\to\langle g,f\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi}g(t)\,f(t)\,dt,\qquad g\in Y.$$

We shall denote this functional also by f.

In what follows, we shall denote for every subset E of $[-\pi,\pi]$, by E^c the set $[-\pi,\pi]\backslash E$.

DEFINITION. If Y is a Banach space of functions on $\mathbb{T}$, then we shall say that it is of class $\mathscr{U}$, if the following conditions hold:

(a) $Y \subset L^2(\mathbb{T})$, and the embedding is continuous.

(b) For every $\varepsilon > 0$, there exists a measurable subset Q of $[-\pi, \pi]$ and a function f in $L^\infty(\mathbb{T})$, such that

$$|Q^c| < \varepsilon, \qquad f = 1 \text{ on } Q, \qquad \text{and} \qquad \|f\|_{Y^*} < \varepsilon.$$

(c) If $g \in Y$, then for every $n \in \mathbb{Z}$, the function

$$t \to e^{int} g(t), \qquad t \in \mathbb{T},$$

is also in Y.

We claim that all the Banach spaces listed in the statement of Theorem 3.6 are of class $\mathscr{U}$. It is clear that they satisfy condition (a) and that $A_q(\mathbb{T})$ also satisfies condition (c). The second hypothesis on the sequence $\omega = \{\omega(n)\}_{n \in \mathbb{Z}}$ implies that the Banach spaces $A_2(\mathbb{T}, \omega)$ also satisfy condition (c). It is proved in [15; 16, Ch. IV, Lemma 2.5], and [20, Lemma 9] that the Banach spaces $A_q(\mathbb{T})$, $1 < q < 2$, satisfy condition (b). To show that this condition holds also for the Banach spaces $A_2(\mathbb{T}, \omega)$, we need an intermediate result.

In what follows, we shall denote for every function h on $\mathbb{T}$, and every positive integer n, by $h(n \cdot)$ the function

$$t \to h(nt), \qquad t \in \mathbb{T}.$$

Lemma 3.7. *Let $\omega = \{\omega(n)\}_{n \in \mathbb{Z}}$ be a sequence of positive numbers such that*

$$\lim_{n \to \pm\infty} \omega(n) = \infty.$$

Set $Y = A_2(\mathbb{T}, \omega)$, and assume that h is a function in $L^2(\mathbb{T})$ such that $c_0(h) = 0$. Then

$$\lim_{n \to \infty} \|h(n \cdot)\|_{Y^*} = 0.$$

Proof. First observe that if f is in $L^2(\mathbb{T})$, then by Parseval's formula

$$\langle g, f \rangle = \sum_{j=-\infty}^{\infty} c_j(g)\, c_{-j}(f), \qquad \forall g \in Y,$$

and therefore,

$$\|f\|_{Y^*}^2 = \sum_{j=-\infty}^{\infty} \frac{|c_j(f)|^2}{\omega(-j)}.$$

Hence setting $b = \inf_{j \in \mathbb{Z}} \omega(j)$, and using the assumption that $c_0(h) = 0$, we obtain that for every k and n in $\mathbb{N}$,

$$\|h(n\cdot)\|^2_{Y^*} \leqslant \sum_{1 \leqslant |j| \leqslant k} \frac{|c_j(h)|^2}{\omega(-nj)} + b^{-1} \sum_{|j| > k} |c_j(h)|^2$$

and consequently from the assumption that

$$\lim_{n \to \pm\infty} \omega(n) = \infty$$

we get that for every k in N,

$$\overline{\lim_{n \to \infty}} \|h(n\cdot)\|^2_{Y^*} \leqslant b^{-1} \sum_{|j| > k} |c_j(h)|^2.$$

Since $h \in L^2(\mathbb{T})$, this implies the desired conclusion. ∎

It is now easy to show that the space $Y = A_2(\mathbb{T}, \omega)$ satisfies condition (b). To this end, assume that $\varepsilon > 0$, and let D be a measurable subset of $[-\pi, \pi]$, such that $0 < |D^c| < \varepsilon$. Set $a = |D|\,|D^c|^{-1}$, and consider the function h in $L^2(\mathbb{T})$ such that

$$h = \chi_D - a\chi_{D^c}, \qquad \text{on } [-\pi, \pi].$$

Then $c_0(h) = 0$, and therefore by Lemma 3.7, there exists a positive integer n such that

$$\|h(n\cdot)\|_{Y^*} < \varepsilon.$$

Observing that the measure of the set

$$Q = \{x \in [-\pi, \pi] : h(nx) = 1\}$$

is $|D|$, we see that condition (b) is satisfied with this set and $f = h(n\cdot)$.

Thus it remains to prove

THEOREM 3.8. *If Y is a Banach space of class $\mathscr{U}$, then for every sequence of positive numbers $\{a_j\}_{j \in \mathbb{N}}$, there exists sequence $\{E_j\}_{j \in \mathbb{N}}$ of mutually disjoint measurable subsets of $[-\pi, \pi]$, which are sets of uniqueness for Y, and*

$$|E_j| \leqslant a_j, \qquad \forall j \in \mathbb{N}.$$

Proof. Let $\{a_j\}_{j \in \mathbb{N}}$ be a sequence of positive numbers. Using condition (b), we define by induction, a sequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ of positive numbers, a sequence $\{Q_n\}_{n \in \mathbb{N}}$ of measurable subsets of $[-\pi, \pi]$, and a sequence

$\{f_n\}_{n\in\mathbb{N}}$ of functions in $L^\infty(\mathbb{T})$, such that for every $n\in\mathbb{N}$, the following conditions hold:

$$|Q_n^c|<\varepsilon, \qquad f_n=1 \text{ on } Q_n, \qquad \|f_n\|_{Y^*}<\varepsilon_n. \tag{1}$$

$$\sum_{j=n+1}^{\infty}\varepsilon_j<\varepsilon_n^2(1+\|f_n\|_\infty)^{-2}, \qquad \text{and} \qquad \varepsilon_n\leqslant\frac{a_j}{n(n+1)},$$
$$\text{for} \quad 1\leqslant j\leqslant n. \tag{2}$$

Let $\{\alpha(n)\}_{n\in\mathbb{N}}$ be a sequence of positive integers such that

$$\alpha(n)\leqslant n, \qquad \forall n\in\mathbb{N},$$

and assume that every j in $\mathbb{N}$ appears in this sequence infinitely many times. (For example, we may define $\alpha(n)=n-2^{k-1}+1$, for $2^{k-1}\leqslant n<2^k$, $k\in\mathbb{N}$.)

We define now, also by induction, a sequence $\{E_j^{(n)}, 1\leqslant j\leqslant n, n\in\mathbb{N}\}$ of measurable subsets of $[-\pi,\pi]$.

Set $E_1^{(1)}=\emptyset$, and assume that for some $n\in\mathbb{N}$, we have already defined measurable subsets $\{E_j^{(n)}, 1\leqslant j\leqslant n\}$ of $[-\pi,\pi]$, which satisfy the conditions

$$|E_j^{(n)}|\leqslant(1-1/n)\,a_j, \qquad 1\leqslant j\leqslant n.$$

Define the sets $\{E_j^{(n+1)}, 1\leqslant j\leqslant n+1\}$ as

$$E_j^{(n+1)}=E_j^{(n)}\cap Q_n, \qquad \text{if} \quad j\leqslant n \quad \text{and} \quad j\neq\alpha(n),$$
$$E_j^{(n+1)}=E_j^{(n)}\cup Q_n^c, \qquad \text{if} \quad j=\alpha(n),$$
$$E_{n+1}^{(n+1)}=\emptyset.$$

It is easily verified that this definition yields a sequence of subsets

$$\{E_j^{(n)}, 1\leqslant j\leqslant n, n\in\mathbb{N}\}$$

of $[-\pi,\pi]$, such that for every n in $\mathbb{N}$, the following conditions hold (below we use the notations introduced in the proof of Proposition 3.3):

The set $E_j^{(n)}$, $1\leqslant j\leqslant n$, are mutually disjoint, and

$$|E_j^{(n)}|\leqslant(1-1/n)\,a_j, \qquad 1\leqslant j\leqslant n. \tag{3}$$

$$Q_n^c\subset E_j^{(n+1)}, \qquad \text{for} \quad j=\alpha(n). \tag{4}$$

$$d(E_j^{(n+1)}, E_j^{(n)})<\varepsilon_n, \qquad 1\leqslant j\leqslant n. \tag{5}$$

It follows from (5) and (2) that if $m > n+1$,

$$d(E_j^{(m)}, E_j^{(n+1)}) < \varepsilon_n^2 (1 + \|f_n\|_\infty)^{-2}, \qquad \text{for} \quad 1 \leqslant j \leqslant n.$$

Hence for every fixed $j \in \mathbb{N}$, the sequence $\{E_j^{(n)}, n = j, j+1, ...\}$ is a Cauchy sequence in $\mathfrak{M}$, and therefore, it converges as $n \to \infty$, to an element E_j in $\mathfrak{M}$, which satisfies the condition

$$d(E_j, E_j^{(n+1)}) \leqslant \varepsilon_n^2 (1 + \|f_n\|_\infty)^{-2}, \qquad \text{for} \quad n \geqslant j.$$

This implies by (4) that

$$|E_j^c \cap Q_n^c| \leqslant \varepsilon_n^2 (1 + \|f_n\|_\infty)^{-2}, \qquad \text{for} \quad j = \alpha(n). \tag{6}$$

We claim that the sequence $\{E_j\}_{j \in \mathbb{N}}$ has all the required properties.

First, it follows from (3) that

$$|E_j \cap E_k| = 0, \qquad \text{for} \quad j \neq k,$$

and

$$|E_j| \leqslant a_j, \qquad \forall j \in \mathbb{N}.$$

To show that the members of this sequence are sets of uniqueness for Y, fix j in $\mathbb{N}$, and assume that g is a function in Y that vanishes on E_j. We prove that $g = 0$ by showing that

$$c_n(g) = 0, \qquad \forall n \in \mathbb{Z}.$$

Since by condition (c) we may replace for every $n \in \mathbb{Z}$ the function g by the function

$$t \to e^{\text{int}} g(t), \qquad t \in \mathbb{T},$$

it suffices to prove that

$$c_0(g) = 0.$$

To show this, fix $n \in \mathbb{N}$, and set $K = E_j^c \cap Q_n^c$. Since by (1) and the assumption on g,

$$g(1 - f_n) = 0, \qquad \text{on} \quad E_j \cup Q_n,$$

we obtain that

$$c_0(g) = \langle g, f_n \rangle + \frac{1}{2\pi} \int_K g(t)(1 - f_n(t))\, dt.$$

Thus, denoting by $\| \ \|_2$ the norm in $L^2(\mathbb{T})$, we get from (1) and the Schwarz inequality that

$$|c_0(g)| \leqslant \|g\|_Y \varepsilon_n + \|g\|_2 (1 + \|f_n\|_\infty) |K|^{1/2}.$$

Therefore, if $\alpha(n)=j$, we obtain from (6) that

$$|c_0(g)| \leqslant (\|g\|_Y + \|g\|_2)\, \varepsilon_n.$$

Since $\alpha(n)=j$ for infinitely many n in $\mathbb{N}$, and $\varepsilon_n \to 0$, this implies that $c_0(g)=0$, and the theorem is proved. ▌

Remark. As mentioned after the statement of Proposition 3.3, Theorem 3.6 implies the proposition. This follows from the fact that the 2π-periodization of every C^∞ function on $\mathbb{R}$ which is supported on an open interval included in $[-\pi, \pi]$ belongs to each of the spaces $A_q(\mathbb{T})$, $1<q<2$, and therefore every measurable subset of $[-\pi, \pi]$ which is a set of uniqueness for one of these spaces has intersection of positive measure with every such interval.

4. RESULTS IN THE OPPOSITE DIRECTION

In this section, we prove some results in the opposite direction and present some necessary conditions for the completeness of integer translates. We first show that a function whose integer translates are complete in one of the spaces $C_0^k(\mathbb{R})$ or $H^{p,k}(\mathbb{R})$, $2<p<\infty$, $k=0, 1, ...,$ must have rather slow decay at infinity. As we have seen, such a function can be chosen in $L^2(\mathbb{R})$, but it turns out that it cannot be chosen in $L^1(\mathbb{R})$. We show that this is true for a more general class of spaces.

Following [16, Ch. VI], we shall call a Banach space X of locally integrable functions on $\mathbb{R}$ homogeneous if it is translation invariant, the translation operators T_s are isometries on X, and for every $g \in X$, the mapping

$$s \to T_s g, \qquad s \in \mathbb{R},$$

from $\mathbb{R}$ into X is continuous.

We note that if X is such a space, then it is invariant under convolutions with elements of $L^1(\mathbb{R})$. More precisely, if $g \in X$ and $f \in L^1(\mathbb{R})$, their convolution $f * g$ (which can be defined as the Bochner integral $\int_{\mathbb{R}} T_s g f(s)\, ds$, of the function $s \to T_s g$, with respect to the measure $f(s)\, ds$) is in X, and

$$\|f * g\|_X \leqslant \|f\|_{L^1(\mathbb{R})} \|g\|_X.$$

In what follows, we denote (in accordance with the notation introduced in Section 3) for every real number t, by M_t the transformation which associates with a function g on $\mathbb{R}$ the function

$$x \to e^{itx} g(x), \qquad x \in \mathbb{R}.$$

THEOREM 4.1. *Let X be a homogeneous Banach space of functions on $\mathbb{R}$, which is not the zero space, and assume that for every $t \in \mathbb{R}$, the transformation M_t maps X continuously into itself. Then the integer translates of a function in $X \cap L^1(\mathbb{R})$ are not complete in X.*

Proof. Assume that φ is in $X \cap L^1(\mathbb{R})$. We shall show that the translates $\{\varphi_n, n \in \mathbb{Z}\}$ are not complete in X. For this, consider the linear transformation $L: X \to X$ defined by

$$Lg = \varphi * M_{2\pi} g - (M_{2\pi} \varphi) * g, \qquad g \in X.$$

By the assumptions on X and the preceding observation, L is continuous. It is easily verified that

$$L(\varphi_n) = 0, \qquad \forall n \in \mathbb{Z}.$$

Hence the assertion will follow if we show that if $\varphi \neq 0$ then $L \neq 0$.

Assume that $L = 0$, then in particular

$$L(M_t \varphi) = 0, \qquad \forall t \in \mathbb{R},$$

and applying the Fourier transform we get that

$$\hat{\varphi}(x)\, \hat{\varphi}(x - 2\pi - t) - \hat{\varphi}(x - t)\, \hat{\varphi}(x - 2\pi) = 0, \qquad \forall x, t \in \mathbb{R},$$

and this means that there exists a constant c such that

$$\hat{\varphi}(x - 2\pi) = c\hat{\varphi}(x), \qquad \forall x \in \mathbb{R}.$$

Since $\hat{\varphi} \in C_0(\mathbb{R})$ (by the assumption that $\varphi \in L^1(\mathbb{R})$) this implies that $\hat{\varphi} = 0$, and therefore $\varphi = 0$. This completes the proof. ∎

Note that the theorem applies in particular to the spaces $C_0^k(\mathbb{R})$ and $H^{p,k}(\mathbb{R})$, $1 \leqslant p < \infty$, $k = 0, 1, \ldots$, since they satisfy its hypotheses.

Remarks. 1. The hypothesis that X is invariant under the operator M_t cannot be omitted from the statement of the theorem. For example, if X is the Paley–Wiener space $L^2(\mathbb{R})_\pi$ of all functions in $L^2(\mathbb{R})$ whose Fourier transform is supported on $[-\pi, \pi]$, then all the other hypotheses of the theorem are satisfied, but X is the closed span of integer translates of any of its functions whose Fourier transform is different from zero a.e. on $[-\pi, \pi]$ (cf. [12, Th. 1]). So in this case, there exists even a function in $\mathscr{S}$ whose integer translates are complete in X.

2. For the spaces $L^p(\mathbb{R})$, $2 < p < \infty$, one can prove a stronger result, namely, if $1 \leqslant s \leqslant p(p-1)^{-1}$, then the integer translates of a function in

$L^s(\mathbb{R}) \cap L^p(\mathbb{R})$ are not complete in $L^p(\mathbb{R})$. This can be proved by an argument which is similar to the one in the proof above, using the fact that if $1/r = 1/s + 1/p - 1$, then by Young's inequality for convolutions,

$$\|f * g\|_{L^r(\mathbb{R})} \leqslant \|f\|_{L^s(\mathbb{R})} \|g\|_{L^p(\mathbb{R})},$$

for every f in $L^s(\mathbb{R})$ and g in $L^p(\mathbb{R})$.

3. By similar arguments, one can prove that if ρ is a tempered weight function on $\mathbb{R}$ such that for some $r \geqslant 0$,

$$\rho(x+y) \leqslant \rho(y)(1+|x|)^r, \qquad x, y \in \mathbb{R},$$

then the integer translates of a function in $L^2(\mathbb{R}, \rho(x)\,dx) \cap L^1(\mathbb{R}, (1+|x|)^r\,dx)$ are not complete in $L^2(\mathbb{R}, \rho(x)\,dx)$.

In view of Theorem 2.1, one may ask for the characterization of functions in $C_0(\mathbb{R})$ and $L^p(\mathbb{R})$, $2 < p < \infty$, whose integer translates are complete. This is a hard problem and seems to be far out of scope. We mention in this connection that for $p \neq 2$, there is no known characterization of functions in $L^p(\mathbb{R})$, $1 < p < \infty$, whose translates are complete. A simple necessary condition is that the support of the Fourier transform of such a function is the whole real line. In fact, it is easy to see that the same is true for every translation invariant Banach space X of tempered distribution which is continuously embedded in $\mathscr{S}'$ (with respect to the ω^*-topology), and the union of the supports of the elements in $\mathscr{F}X$ is $\mathbb{R}$. Hence this is in particular also a necessary condition for completeness of integer translates of an element in such a space.

It follows from these observations and the Paley–Wiener–Schwartz theorem that the function φ in Theorem 2.1 cannot be chosen to be the restriction to $\mathbb{R}$ of an entire function of finite exponential type.

We show next that each of the spaces $H^{p,k}(\mathbb{R})$, $1 \leqslant p \leqslant 2$, $k = 0, 1, ...$, is not the closed span of integer translates of finitely many elements. This is a particular case of:

THEOREM 4.2. *Let X be a translation invariant Banach space of tempered distributions on $\mathbb{R}$, which is mapped continuously by the Fourier transform into the Fréchet space $L^1_{\mathrm{loc}}(\mathbb{R})$, and assume that X contains an element whose Fourier transform is different from zero a.e. Then X is not the closed span of integer translates of finitely many elements.*

Before giving the proof, we make some observations. We note first that the spaces $H^{p,k}(\mathbb{R})$, $1 \leqslant p \leqslant 2$, $k = 0, 1, ...$, satisfy the hypotheses of the theorem. This is clear for the first and the third hypothesis, and the second follows from the Hausdorff–Young theorem.

The assertion of the theorem for the space $L^2(\mathbb{R})$ is known and can be deduced from the multiplicity theory for unitary operators on Hilbert space and also from the results in [6].

It is easily verified that the heuristic argument described in the first paragraph of the introduction is actually correct for the spaces which satisfy the hypotheses of the theorem, and thus provides a proof of the fact that such a space is not singly generated. The general case can be deduced from Proposition 2.1 in [3]. For the sake of completeness we adapt it to our setting, and present a direct proof.

Proof of Theorem 4.2. Let J be a finite subst of X, and denote by X_0 the linear span (in the algebraic sense) of integer translates of all elements of J. We claim that $\bar{X}_0 \neq X$. Let m denote the number of elements of J, and set $n = m + 1$. Denote by $\mathscr{A}$ the algebra of all measurable functions on $\mathbb{R}$, and consider the n-linear mapping Ψ: $X^n \to \mathscr{A}$, defined by

$$\Psi(v^1, v^2, ..., v^n) = \det(T_{2\pi(k-1)} \hat{v}^j)_{j,k=1}^n,$$

where $(v^1, v^2, ..., v^n)$ is an n-tuple in X^n. The claim will be proved by showing that Ψ annihilates $\bar{X}_0^n$ and $\Psi \neq 0$. We first show that Ψ annihilates X_0^n. By linearity, it suffices to prove that Ψ annihilates every n-tuple which consists of integer translates of elements in J. Assume that $(w^1, w^2, ..., w^n)$ is such an n-tuple. Since J contains m elements, and $n = m + 1$, there exist two members of this n-tuple, which are integer translates of the same element, say v, of J. By changing the order, we may assume that these are the first and second members; that is, we may assume that

$$(w^1, w^2, ..., w^n) = (T_p v, T_q v, ..., w^n)$$

for some integers p and q. But this implies by the definition of Ψ that

$$\Psi(w^1, w^2, ..., w^n) = M_{-p-q} \Psi(v, v, ..., w^n) = 0.$$

Thus Ψ annihilates X_0^n. Since the Fourier transform maps X continuously into the Fréchet space $L^1_{\text{loc}}(\mathbb{R})$, every convergent sequence in X has a subsequence which is mapped by the Fourier transform into a sequence of measurable functions which converges a.e. on $\mathbb{R}$, and therefore Ψ also annihilates $\bar{X}_0^n$.

It remains to show that $\Psi \neq 0$; To this end, assume that u is an element in X such that $\hat{u} \neq 0$, a.e., and consider the n-tuple $(u^1, u^2, ..., u^n)$ in X^n defined by

$$u^j = T_{j/2\pi} u, \qquad j = 1, 2, ..., n.$$

We obtain from the definition of Ψ and the formula for Vandermonde determinants that

$$\Psi(u^1, u^2, ..., u^n) = ah,$$

where

$$a = \prod_{1 \leqslant j < k < n} (e^{ik} - e^{ij}),$$

and h is the function on $\mathbb{R}$ defined by

$$h(x) = \exp\left[-\frac{i}{4\pi} n(n+1)\, x \right] \cdot \prod_{j=0}^{n-1} \hat{u}(x - 2\pi j), \qquad x \in \mathbb{R}.$$

The assumption on u implies that $h \neq 0$ a.e., thus $\Psi \neq 0$, and the theorem is proved. ∎

We conclude this section with some observations about Schauder bases. Since each of the Banach spaces listed in the statement of Theorem 2.1 contains a function whose integer translates are complete and also form a minimal system, it is natural to ask whether they also possess a Schauder basis which consists of integer translates of a single function. Unfortunately, this is not the case. We consider this problem first in a more general setting.

In what follows, we assume that X is a translation invariant Banach space of tempered distributions on $\mathbb{R}$, on which the translation operators T_s, $s \in \mathbb{R}$, are continuous, that X includes the Schwartz space $\mathscr{S}$, and that the embedding is continuous and dense. Then the adjoint of this embedding embeds the dual space X^* continuously into $\mathscr{S}'$ (with respect to the w^* topology), and the adjoint of the operator T_s on X is the operator T_{-s} on X^*. Thus X^* is also a translation invariant Banach space of tempered distributions, and for $u \in X$ and $v \in X^*$,

$$\langle u_s, v_t \rangle = \langle u_{s-t}, v \rangle, \qquad \forall s, t \in \mathbb{R}.$$

Asume that u is an element of X, whose integer translates form a minimal system, and let v be an element of X^* such that

$$\langle u_n, v \rangle = \delta_{n,0}, \qquad \forall n \in \mathbb{Z}.$$

Then by the preceding observation,

$$\langle u_n, v_k \rangle = \langle u_{n-k}, v \rangle = \delta_{n,k}, \qquad \forall n \in \mathbb{Z},$$

so that $\{v_k, k \in \mathbb{Z}\}$ is a biorthogonal sequence for $\{u_n, n \in \mathbb{Z}\}$.

Assume now that the sequence $\{u_n, n \in \mathbb{Z}\}$, ordered properly, is a Schauder basis for X. Then the biorthogonal sequence $\{v_n, n \in \mathbb{Z}\}$ is total; that is, its linear span is w^*-dense in X^* (cf. [18, Ch. I]).

If B is a Banach space of class $\mathscr{H}$, then we know that the Fourier transform maps B^* continuously into the Fréchet space $L^1_{\mathrm{loc}}(\mathbb{R})$. Therefore if B^* contains an element whose Fourier transform is different from zero a.e., we obtain from Theorem 4.2 that it is not singly generated. Hence if B is reflexive, it follows from the preceding observations that it does not possess a Schauder basis which consists of integer translates of a single function. Consequently, the Banach spaces $H^{p,k}(\mathbb{R})$, $1 < p < \infty$, $k = 0, 1, ...,$ and $L^2(\mathbb{R}, \rho(x)\, dx)$ (where ρ is a tempered weight function in $C_0(\mathbb{R})$) do not possess such a basis.

To prove the assertion for the non-reflexive spaces $C_0^k(\mathbb{R})$, $k = 0, 1, ...,$ we have to show that their duals do not contain an element such that the linear span of its integer translates is w^*-dense. For $M(\mathbb{R})$, the dual space of $C_0(\mathbb{R})$, this can be proved by an argument that is similar to the one in the proof of Theorem 4.1. We give a brief outline.

Let $\mu \in M(\mathbb{R})$, and assume that $\mu \neq 0$. Consider the linear transformation $L: M(\mathbb{R}) \to M(\mathbb{R})$ defined by

$$L\nu = \mu * M_{2\pi}\nu - \nu * M_{2\pi}\mu, \qquad \nu \in M(\mathbb{R}).$$

It is easily verified that L is continuous with respect to the w^* topology of $M(\mathbb{R})$, that it annihilates the integer translates of μ, and that $L \neq 0$. This shows that the linear span of the integer translates of μ is not w^*-dense in $M(\mathbb{R})$.

For the spaces $M^{-k}(\mathbb{R})$, the dual spaces of $C_0^k(\mathbb{R})$, $k = 1, 2, ...,$ the assertion is proved in a similar way, by observing that for every $0 \neq u \in M^{-k}(\mathbb{R})$, the transformation

$$v \to u * M_{2\pi}v - v * M_{2\pi}u, \qquad v \in M^{-k}(\mathbb{R}),$$

maps $M^{-k}(\mathbb{R})$ into $M^{-2k}(\mathbb{R})$, is continuous with respect to the w^* topologies of these spaces, annihilates the integer translates of u, and is not the zero map.

Finally we note that since each of the Banach spaces described in the statement of Theorem 2.1 has a Schauder basis (cf. [19, Ch. VI]), it follows from that theorem and the Krein–Milman–Rutman theorem (cf. [18, Proposition 1.a.9]) that each of these spaces also possesses a Schauder basis whose members are linear combinations of integer translates of a single function.

5. FUNCTION SPACES ON $\mathbb{R}^N$ AND [0, 1]

All the preceding results can be extended, with essentially the same proofs, to the multivariate setting. One can also get directly from Theorem 2.1 the existence of complete sequences which consist of $\mathbb{Z}^n$-translates (or even $\mathbb{Z}^n_+$-translates) of a single function, in each of the Banach spaces $C_0(\mathbb{R}^n)$, $L^p(\mathbb{R}^n)$, $2<p<\infty$, and the corresponding Sobolev spaces. This can be achieved by forming the n-fold tensor product of the corresponding function φ in Theorem 2.1, since each of these Banach spaces is the closed linear span of tensor products of functions in $\mathscr{D}(\mathbb{R})$, and the norm of such a tensor product is the product of the norms of its factors in the corresponding univariate space.

One can also deduce from Theorem 2.1 the corresponding result for the spaces $L^2(\mathbb{R}^n, \rho(x)\,dx)$, where ρ is a tempered weight function on $\mathbb{R}^n$ which is in $C_0(\mathbb{R}^n)$, but this requires a preliminary argument, which reduces the general case to the case where ρ is a tensor product of univariate weight functions of the same type. We omit the details.

Finally, we use Theorem 2.1 to construct certain complete sequences in some Banach spaces of functions on $[0, 1]$. In what follows we shall denote by L the linear transformation which acts on functions f on $[0, 1]$ by

$$Lf(t) = f(t^2), \qquad t \in [0, 1].$$

We denote by $C_0[0, 1]$ the Banach space of all continuous functions on $[0, 1]$, which vanish at 0 and 1, equipped with the maximum norm.

THEOREM 5.1. *Each of the Banach spaces $C_0[0, 1]$ and $L^p[0, 1]$, $1 \leqslant p < \infty$, contains a function f such that the sequence iterates $\{L^n f, n=0, ...\}$ is complete.*

Proof. Since $C_0[0, 1]$ is dense in each of the spaces listed above and the embedding is continuous, it suffices to prove the theorem for this space. Consider the function $v\colon \mathbb{R} \to [0, 1]$ defined by

$$v(x) = \exp(-2^{-x}), \qquad x \in \mathbb{R},$$

and denote by V the linear transformation which associates with a function g on $[0, 1]$ the function $g \circ v$ on $\mathbb{R}$. Then V maps the Banach space $C_0[0, 1]$ isometrically onto the Banach space $C_0(\mathbb{R})$, and a simple computation shows that

$$VL^n = T_n V, \qquad n = 0, 1, ...,$$

where T_n denotes as before the operator of translation by n. Therefore, choosing (by Theorem 2.1) a function φ in $C_0(\mathbb{R})$ whose non-negative integer translates are complete in this space, and setting $f = V^{-1}\varphi$, we obtain that the sequence $\{L^n f, n = 0, 1, ...\}$ is complete in $C_0[0, 1]$. ∎

Remark. Since the function φ in the proof above can be chosen by Theorem 2.1 to be the restriction to $\mathbb{R}$ of an entire function, we see that the function f above can be chosen to be infinitely differentiable on the open interval $(0, 1)$. However, it cannot satisfy a Hölder condition of any order at the end points, and more generally, the function g on $(0, 1)$ defined by

$$g(t)=t^{-1}(1-t)^{-1} f(t), \qquad 0<t<1,$$

cannot be in $L^1[0, 1]$. This can be seen as follows. The proof of Theorem 5.1 shows that if the sequence $\{L^n f,\ n=0, 1, ...\}$ is complete in $C_0[0, 1]$, then the sequence $\{T_n Vf,\ n=0, 1, ...\}$ is complete in $C_0(\mathbb{R})$, and therefore, by Theorem 4.1, Vf cannot be in $L^1(\mathbb{R})$. But it is easily verified that if g is in $L^1[0, 1]$, then Vf is in $L^1(\mathbb{R})$.

Note added in proof. We are grateful to Professor N. K. Nikolskii, who after reading a preprint of this paper informed us that the fact that the spaces $L^p(\mathbb{R})$, $2<p<\infty$, and $C_0(\mathbb{R})$ are singly generated can be also deduced from his results which appear in the book "Selected Problems in Weighted Approximation and Spectral Analysis," Proceedings of the Steklov Institute of Mathematics 120 (1974); English translation: Amer. Math. Soc., Providence, RI, 1976. More specifically, this deduction can be made by using Theorem 1, Lemma 6, and Theorem 6 in Section 3.4 of that book, with $X=l^p(\mathbb{Z})$ and $c_0(\mathbb{Z})$ and $E=L^p[0, 1]$ and $C[0, 1]$, respectively. He also informed us that Theorem 5.1 in this paper was also proved in the Ph.D. thesis of A. K. Kitover in 1973.

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